

The Complex Structures on S^{2n}

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Abstract

In this paper, we show that the twistor space $\mathcal{J}(R^{2n+2})$ on Euclidean space R^{2n+2} is a Kaehler manifold and the orthogonal twistor space $\tilde{\mathcal{J}}(S^{2n})$ of the sphere S^{2n} is a Kaehler submanifold of $\mathcal{J}(R^{2n+2})$. Then we show that an orthogonal almost complex structure J_f on S^{2n} is integrable if and only if the corresponding section $f: S^{2n} \rightarrow \tilde{\mathcal{J}}(S^{2n})$ is holomorphic. These shows there is no integrable orthogonal complex structure on the sphere S^{2n} for $n > 1$.

1. Introduction

An almost complex structure J on a differentiable manifold is an endomorphism $J: TM \rightarrow TM$ of the tangent bundle such that $J^2 = -1$. It is known [2] that the only sphere admitting such structures are S^2 and S^6 . These can also be proved as follows.

If the sphere S^{2n} has an almost complex structure, there is a complex vector bundle $E = T^{(1,0)}S^{2n}$ and there is a twisted Signature operator $D: \Gamma(\wedge^+(S^{2n}) \otimes E) \rightarrow \Gamma(\wedge^-(S^{2n}) \otimes E)$. The index of operator D is (see [5], p. 256, Theorem 13.9)

$$\text{ind}(D) = \int_{S^{2n}} L(TS^{2n}) \cdot \sum_j 2^j ch_j(E) = \int_{S^{2n}} 2^n ch_n(E),$$

where Hirzebruch L-class $L(TS^{2n}) = 1$. Let $c(E) = \sum_{i=0}^n c_i(E) = \prod_{i=1}^n (1 + x_i)$ be the total Chern class of E , x_i the Chern roots, $c_1(E) = \cdots = c_{n-1}(E) = 0$, $c_n(E) =$

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$e(TS^{2n})$ be the Euler class of S^{2n} (see [12]). Note that

$$\begin{aligned} n!ch_n(E) &= x_1^n + \cdots + x_n^n = \sum_i x_i^{n-1} \cdot \sum_j x_j - \sum_{i \neq j} x_i^{n-1} \cdot x_j \\ &= (n-1)!ch_{n-1}(E)c_1(E) - \sum_{i \neq j} x_i^{n-1} \cdot x_j, \end{aligned}$$

$$\sum_{i \neq j} x_i^{n-1} \cdot x_j = \frac{1}{2} \sum_i x_i^{n-2} \cdot \sum_{j \neq k} x_j x_k - \frac{1}{2} \sum_{i,j,k} x_i^{n-2} x_j x_k,$$

where i, j, k are all different. These shows

$$ch_n(E) = \frac{(-1)^{n-1}}{(n-1)!} c_n(E).$$

Then

$$\text{ind}(D) = \int_{S^{2n}} \frac{(-1)^{n-1} 2^n}{(n-1)!} c_n(E) = \frac{(-2)^{n+1}}{(n-1)!}.$$

This shows that there are no almost complex structure on S^{2n} when $n > 3$.

If there is an almost complex structure on S^4 , we have a Dolbeault operator $\bar{D}: A^{(0, \text{even})}(S^4) \rightarrow A^{(0, \text{odd})}(S^4)$. The index of operator \bar{D} is

$$\text{ind}(\bar{D}) = \int_{S^4} Td(T^{(1,0)}S^4),$$

where Todd class $Td(T^{(1,0)}S^4) = 1 + \frac{1}{12}c_2(T^{(1,0)}S^4) = 1 + \frac{1}{12}e(S^4)$. Hence we have $\text{ind}(\bar{D}) = \frac{1}{6}$, this shows there is no almost complex structure on S^4 .

The structure $J: TM \rightarrow TM$ is said to be integrable if it comes from an honest complex structure on M , then M is a complex manifold. It is a long-standing problem that whether there is an integrable complex structure on the sphere S^6 . In 1986 Hsiung [3] proposed a proof of the nonexistence of an orthogonal complex structure on S^6 . In 1995, he published a book [4] on this subject. There are many computation in Hsiung's proof, I can not repeat his proof. In 1987 Lebrun [6] studied the same problem. In this short paper, he proved that $\nabla_{X_\alpha} X_\beta = -\nabla_{X_\beta} X_\alpha$ for $(0, 1)$ vector fields if the orthogonal complex structure J is integrable. Then he claimed that this can be used to prove that the map $\tau: U \rightarrow G_3(C^7)$ is a holomorphic map. In this paper we shall show that $\nabla_{X_\alpha} X_\beta = -\nabla_{X_\beta} X_\alpha$ can not hold, see Lemma 4.2 below.

In [9], we use Clifford algebra and the spinor calculus to study the orthogonal complex structures on Euclidean space R^8 and the spheres S^4, S^6 . By the spin representation we show that the Grassmann manifold $G(2, 8)$ can be viewed as the set of orthogonal complex structures on R^8 . There are two fibre bundles $\tau: G(2, 8) \rightarrow S^6$

and $\tau_1: CP^3 \rightarrow S^4$ defined naturally. In this way, we show that $G(2, 8)$ and CP^3 can be looked as twistor spaces of S^6 and S^4 respectively. Then we show that there is no orthogonal complex structure on the sphere S^6 .

In this paper we study the general problem. Let $\mathcal{J}(R^{2n})$ be the set of all complex structures on Euclidean space R^{2n} . We first show that, with naturally defined metric and complex structure the twistor space $\mathcal{J}(R^{2n})$ is a Kaehler manifold. Then we show that the orthogonal twistor space $\widetilde{\mathcal{J}}(S^{2n})$ is a Kaehler submanifold of $\mathcal{J}(R^{2n+2})$, where $\widetilde{\mathcal{J}}(S^{2n})$ is the set of orthogonal complex structures on S^{2n} . While, the twistor space $\mathcal{J}(S^{2n})$ of the sphere S^{2n} formed by all complex structures is not a Kaehler submanifold of $\mathcal{J}(R^{2n+2})$. It is known that any almost complex structure on S^{2n} can be determined by a section f of the fibre bundle $\pi: \mathcal{J}(S^{2n}) \rightarrow S^{2n}$.

In Theorem 3.4, we show that an orthogonal almost complex structure on S^{2n} is integrable if and only if the map $f: S^{2n} \rightarrow \widetilde{\mathcal{J}}(S^{2n})$ is holomorphic. Then S^{2n} is a Kaehler manifold if there is an orthogonal integrable complex structure on it. These shows there is no orthogonal integrable complex structure on the sphere S^{2n} for $n > 1$.

Let $f: S^{2n} \rightarrow \mathcal{J}(S^{2n}) \subset \mathcal{J}(R^{2n+2})$ be a section which may be defined locally. In §4 we show that the map $f: S^{2n} \rightarrow \mathcal{J}(R^{2n+2})$ is holomorphic if and only if the complex structure defined by f is orthogonal and integrable.

There are two connected components on $\mathcal{J}(S^{2n})$. Denote $\mathcal{J}^+(S^{2n})$ the twistor space of all oriented complex structures on S^{2n} and $\widetilde{\mathcal{J}}^+(S^{2n})$ the subspace of orthogonal complex structures. The space $\widetilde{\mathcal{J}}^+(S^{2n})$ is a deformation retract of $\mathcal{J}^+(S^{2n})$ and the cohomology groups of these two spaces are the same. Theorem 3.6 shows that the Poincaré polynomial of $\widetilde{\mathcal{J}}^+(S^{2n})$ is

$$P_t(\widetilde{\mathcal{J}}^+(S^{2n})) = (1 + t^2)(1 + t^4) \cdots (1 + t^{2n}).$$

The method used in this paper is differential geometry, especially, the moving frame and the Riemannian connection.

2. The twistor space $\mathcal{J}(R^{2n})$

Let $\bar{e}_i = (0, \dots, 0, 1, 0, \dots, 0)^t$ ($i = 1, \dots, 2n$) be the basis of Euclidean space R^{2n} and $GL(2n, R)$ be the general linear group. Let $J_0 = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & \ddots & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$ be a complex structure on R^{2n} , $J_0 \bar{e}_{2i-1} = \bar{e}_{2i}$, $J_0 \bar{e}_{2i} = -\bar{e}_{2i-1}$. Let $GL(n, C)$ be the

stability subgroup of J_0 :

$$GL(n, C) = \{g \in GL(2n, R) \mid gJ_0 = J_0g\}.$$

Then

$$\mathcal{J}(R^{2n}) = \frac{GL(2n, R)}{GL(n, C)} = \{gJ_0g^{-1} \mid g \in GL(2n, R)\}$$

is the set of all complex structures on R^{2n} called a twistor space on R^{2n} . It is clear we have

$$\mathcal{J}(R^{2n}) = \{A \in GL(2n, R) \mid A^2 = -I\}.$$

Lemma 2.1 For any $A = gJ_0g^{-1} \in \mathcal{J}(R^{2n})$, $g = (e_1, e_2, \dots, e_{2n})$, where $e_l \in R^{2n}$ are column vectors, we have $Ae_{2i-1} = e_{2i}$, $Ae_{2i} = -e_{2i-1}$.

Proof The Lemma follows from

$$A(e_1, e_2, \dots, e_{2n}) = (e_1, e_2, \dots, e_{2n})J_0 = (e_2, -e_1, \dots, e_{2n}, -e_{2n-1}). \quad \square$$

For any $B \in GL(2n, R)$, define inner product on $T_B GL(2n, R) = gl(2n, R)$ by

$$\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY^t) + \frac{1}{2} \text{tr}(BXB^{-1}(BYB^{-1})^t), \quad X, Y \in gl(2n, R).$$

Restricting this inner product on $T\mathcal{J}(R^{2n})$ makes $\mathcal{J}(R^{2n})$ a Riemannian manifold.

It is easy to see that

$$\begin{aligned} T_A \mathcal{J}(R^{2n}) &= \{X \in gl(2n, R) \mid AX + XA = 0\} \\ &= \{X \in gl(2n, R) \mid X = AXA\}. \end{aligned}$$

The normal space of $T_A \mathcal{J}(R^{2n})$ in $T_A GL(2n, R)$ with inner product $\langle \cdot, \cdot \rangle$ is

$$\begin{aligned} T_A^\perp \mathcal{J}(R^{2n}) &= \{Y \in gl(2n, R) \mid AY - YA = 0\} \\ &= \{Y \in gl(2n, R) \mid Y = -AYA\}. \end{aligned}$$

The spaces $T_A \mathcal{J}(R^{2n})$ and $T_A^\perp \mathcal{J}(R^{2n})$ are all invariant by the action $Ad(A)$. Then for any $X_1, X_2 \in T_A \mathcal{J}(R^{2n})$, we have

$$\langle X_1, X_2 \rangle = \text{tr}(X_1 X_2^t).$$

Similarly, for any $Y_1, Y_2 \in T_A^\perp \mathcal{J}(R^{2n})$, we have $\langle Y_1, Y_2 \rangle = \text{tr}(Y_1 Y_2^t)$.

For any $A \in \mathcal{J}(R^{2n})$, the map $X \in gl(2n, R) \mapsto AX$ defines a complex structure on $gl(2n, R)$. It is easy to see that

$$A: T_A \mathcal{J}(R^{2n}) \rightarrow T_A \mathcal{J}(R^{2n}), \quad A: T_A^\perp \mathcal{J}(R^{2n}) \rightarrow T_A^\perp \mathcal{J}(R^{2n}).$$

The maps $A: T_A\mathcal{J}(R^{2n}) \rightarrow T_A\mathcal{J}(R^{2n})$ define an almost complex structure \tilde{J} on $\mathcal{J}(R^{2n})$. In general,

$$\langle X_1, X_2 \rangle \neq \langle \tilde{J}X_1, \tilde{J}X_2 \rangle, \quad X_1, X_2 \in T_A\mathcal{J}(R^{2n}).$$

Let \widetilde{D} be the Riemannian connection on $\mathcal{J}(R^{2n})$ with respect to the Riemannian metric defined above.

Lemma 2.2 $\widetilde{D}\tilde{J} = 0$.

Proof Let X_1, \dots, X_{2n^2} be tangent frame fields on $\mathcal{J}(R^{2n})$ such that $\tilde{J}X_{2i-1} = X_{2i}$, $\tilde{J}X_{2i} = -X_{2i-1}$. Let Y_α be local sections on $T^\perp\mathcal{J}(R^{2n})$. Then we can write

$$dX_l = \sum \omega_l^k X_k + \sum \omega_l^\alpha Y_\alpha.$$

By Gauss formula, the connection \widetilde{D} is determined by

$$\widetilde{D}X_l = \sum \omega_l^k X_k.$$

Let $dA = \sum \omega^l X_l$. From $\tilde{J}X_{2i-1} = AX_{2i-1} = X_{2i}$, we have

$$\begin{aligned} dX_{2i} &= \sum \omega_{2i}^k X_k + \sum \omega_{2i}^\alpha Y_\alpha \\ &= \sum \omega^l X_l X_{2i-1} + \sum \omega_{2i-1}^k AX_k + \sum \omega_{2i-1}^\alpha AY_\alpha. \end{aligned}$$

By $AX_l X_{2i-1} = X_l X_{2i-1} A$, we know that $X_l X_{2i-1} \in T^\perp\mathcal{J}(R^{2n})$, hence $\sum \omega_{2i}^k X_k = \sum \omega_{2i-1}^k AX_k$ and we have

$$\omega_{2i-1}^{2j-1} = \omega_{2i}^{2j}, \quad \omega_{2i}^{2j-1} = -\omega_{2i-1}^{2j}.$$

Then $\widetilde{D}(\tilde{J}X_{2i-1}) = \tilde{J}\widetilde{D}X_{2i-1}$ and

$$(\widetilde{D}\tilde{J})X_{2i-1} = \widetilde{D}(\tilde{J}X_{2i-1}) - \tilde{J}\widetilde{D}X_{2i-1} = 0.$$

Similarly, $(\widetilde{D}\tilde{J})X_{2i} = 0$. \square

For any $X, X' \in T\mathcal{J}(R^{2n})$ define

$$ds^2(X, X') = \frac{1}{2}\langle X, X' \rangle + \frac{1}{2}\langle \tilde{J}X, \tilde{J}X' \rangle.$$

The almost complex structure \tilde{J} is orthogonal with respect to the metric ds^2 .

Theorem 2.3 With Riemannian metric ds^2 and the complex structure \tilde{J} , the twistor space $\mathcal{J}(R^{2n})$ is a Kaehler manifold.

Proof By Lemma 2.2, \widetilde{D} is the Riemannian connection with respect to the metric ds^2 . $\mathcal{J}(R^{2n})$ is an almost Hermitian manifold with metric ds^2 and almost

complex structure \tilde{J} . From $\tilde{D}\tilde{J} = 0$, we know that \tilde{J} is integrable and $\mathcal{J}(R^{2n})$ is a Kaehler manifold. \square

The orthogonal twistor space $\tilde{\mathcal{J}}(R^{2n}) = O(2n)/U(n)$ on Euclidean space R^{2n} is a subspace of $\mathcal{J}(R^{2n})$. Note that the induced metric on $\tilde{\mathcal{J}}(R^{2n})$ is

$$ds^2(X, X') = ds^2(\tilde{J}X, \tilde{J}X') = \langle X, X' \rangle = \text{tr}(XX'^t).$$

Theorem 2.4 (1) $\tilde{\mathcal{J}}(R^{2n})$ is a Kaehler submanifold of $\mathcal{J}(R^{2n})$;

(2) $\tilde{\mathcal{J}}(R^{2n})$ is a deformation retract of $\mathcal{J}(R^{2n})$.

Proof $\tilde{\mathcal{J}}(R^{2n}) = \{A \in \mathcal{J}(R^{2n}) \mid A^t = -A\}$ is a submanifold of $\mathcal{J}(R^{2n})$ and the tangent space $T_A\tilde{\mathcal{J}}(R^{2n})$ is invariant under the action \tilde{J} . Then $\tilde{\mathcal{J}}(R^{2n})$ is complex submanifold of $\mathcal{J}(R^{2n})$, hence a Kaehler submanifold.

For (2), let A be any element of $\mathcal{J}(R^{2n})$, set $A = A_1 + A_2$ where $A_1 = -A_1^t$, $A_2 = A_2^t$. From $A^2 = A_1^2 + A_2^2 + A_1A_2 + A_2A_1 = -I$ and $(A_1A_2 + A_2A_1)^t = -A_1A_2 - A_2A_1$, we have

$$A_1^2 + A_2^2 = -I, \quad A_1A_2 + A_2A_1 = 0.$$

It is easy to see that A_1 is a non-singular matrix. Let e_1, \dots, e_{2n} be orthonormal vectors such that $A_2e_i = \lambda_i e_i$, then $A_2 = \sum \lambda_i e_i \cdot e_i^t$. Let $A_1 = BP$ be the unique decomposition such that $B \in \tilde{\mathcal{J}}(R^{2n})$, $BP = PB$, $P^2 = -A_1^2$ and P is positive definite, for proof see [8], p.205,213. Hence $P = \sum \sqrt{1 + \lambda_i^2} e_i \cdot e_i^t$, $PA_2 = A_2P$, $BA_2 + A_2B = 0$. The element A is determined by B and A_2 with $BA_2 + A_2B = 0$.

As $A_2Be_i = -BA_2e_i = -\lambda_i e_i$, then if λ_i is a non-zero characteristic value of A_2 , $-\lambda_i$ is also a characteristic value of A_2 with characteristic vector Be_i , $B(Be_i) = -e_i$. These shows that for any $A \in \mathcal{J}(R^{2n})$, there are orthonormal vectors e_i, e_{n+i} and numbers λ_i such that $A = A_1 + A_2 = BP + A_2$ with

$$A_2 = \sum_{i=1}^n \lambda_i (e_i e_i^t - e_{n+i} e_{n+i}^t), \quad P = \sum_{i=1}^n \sqrt{1 + \lambda_i^2} (e_i e_i^t + e_{n+i} e_{n+i}^t),$$

$$B = \sum_{i=1}^n (e_{n+i} e_i^t - e_i e_{n+i}^t),$$

$$A = \sum_{i=1}^n \sqrt{1 + \lambda_i^2} (e_{n+i} e_i^t - e_i e_{n+i}^t) + \sum_{i=1}^n \lambda_i (e_i e_i^t - e_{n+i} e_{n+i}^t).$$

Let $A_2(t) = tA_2$, $P(t) = \sum \sqrt{1 + t^2 \lambda_i^2} (e_i e_i^t + e_{n+i} e_{n+i}^t)$. Then $A(t) = BP(t) + A_2(t) \in \mathcal{J}(R^{2n})$. These shows $\tilde{\mathcal{J}}(R^{2n})$ is a deformation retract of $\mathcal{J}(R^{2n})$.

Then we have a map $\mathcal{J}(R^{2n}) \rightarrow \tilde{\mathcal{J}}(R^{2n})$ defined by $A \mapsto B$. \square

3. The orthogonal complex structure on the sphere S^{2n}

Let $g = (e_{-1}, e_0, e_1, \dots, e_{2n}) \in GL(2n+2, R)$ and $g^{-1} = \begin{pmatrix} s_{-1} \\ s_0 \\ \vdots \\ s_{2n} \end{pmatrix}$, where e_{-1}, \dots, e_{2n} are column vectors and s_{-1}, \dots, s_{2n} are row vectors. Then

$$A = gJ_0g^{-1} = \sum_{i=0}^n (e_{2i} \cdot s_{2i-1} - e_{2i-1} \cdot s_{2i}).$$

Let $e_{-1} = (1, 0, \dots, 0)^t \in R^{2n+2}$ be a fixed vector and

$$S^{2n} = \{e_0 \in R^{2n+2} \mid |e_0| = 1, e_0 \perp e_{-1}\}.$$

The twistor space $\mathcal{J}(S^{2n})$ on S^{2n} can be represented by

$$\mathcal{J}(S^{2n}) = \{A \in \mathcal{J}(R^{2n+2}) \mid Ae_{-1} \in S^{2n} \text{ and } Av \perp e_{-1}, Ae_{-1} \text{ for } v \perp e_{-1}, Ae_{-1}\}.$$

The projection $\pi: \mathcal{J}(S^{2n}) \rightarrow S^{2n}$ is $\pi(A) = Ae_{-1}$. $\mathcal{J}(S^{2n})$ is a Riemannian submanifold of $\mathcal{J}(R^{2n+2})$ with the induced metric.

Let $\widetilde{\mathcal{J}}(S^{2n})$ be a subspace of $\mathcal{J}(S^{2n})$ formed by all orthogonal complex structures on S^{2n} . By Theorem 2.4, $\widetilde{\mathcal{J}}(S^{2n})$ is a deformation retract of $\mathcal{J}(S^{2n})$. Then any almost complex structure on S^{2n} can be deformed to an orthogonal almost complex structure.

For any $A = gJ_0g^{-1} \in \widetilde{\mathcal{J}}(R^{2n+2})$, $g \in O(2n+2)$, we can choose an $h \in U(n+1) \subset SO(2n+2)$ such that $gh = (e_{-1}, e_0, e_1, \dots, e_{2n})$, where $e_{-1} = (1, 0, \dots, 0)$. Then

$$A = gJ_0g^t = e_0e_{-1}^t - e_{-1}e_0^t + \sum_{i=1}^n (e_{2i}e_{2i-1}^t - e_{2i-1}e_{2i}^t).$$

$\sum_{i=1}^n (e_{2i}e_{2i-1}^t - e_{2i-1}e_{2i}^t)$ defines an orthogonal almost complex structure on tangent space $T_{e_0}(S^{2n})$. These shows $\widetilde{\mathcal{J}}(S^{2n}) = \widetilde{\mathcal{J}}(R^{2n+2})$, see also [5].

Theorem 3.1 $\widetilde{\mathcal{J}}(S^{2n})$ is a Kaehler submanifold of $\mathcal{J}(R^{2n+2})$.

Proof The theorem follows from Theorem 2.4. In the following we give a direct proof. We need only to show that for any $A \in \widetilde{\mathcal{J}}(S^{2n})$, $\tilde{J}(T_A\widetilde{\mathcal{J}}(S^{2n})) = T_A\widetilde{\mathcal{J}}(S^{2n})$, where \tilde{J} is the complex structure on $\mathcal{J}(R^{2n+2})$ defined in section 2.

The elements of $\widetilde{\mathcal{J}}(S^{2n})$ can be represented by $A = \sum_{i=0}^n (e_{2i} \cdot e_{2i-1}^t - e_{2i-1} \cdot e_{2i}^t)$, where $e_{-1} = (1, 0, \dots, 0)^t$, $e_0 \in S^{2n}$, $(e_{-1}, e_0, e_1, \dots, e_{2n}) \in O(2n+2)$. By the

method of moving frame we have

$$d(e_0, e_1, \dots, e_{2n}) = (e_0, e_1, \dots, e_{2n}) \begin{pmatrix} 0 & -\omega^1 & -\omega^2 & \dots & -\omega^{2n} \\ \omega^1 & 0 & -\omega_{12} & \dots & -\omega_{1,2n} \\ \omega^2 & \omega_{12} & 0 & \dots & -\omega_{2,2n} \\ \omega^3 & \omega_{13} & \omega_{23} & \dots & -\omega_{3,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{2n} & \omega_{1,2n} & \omega_{2,2n} & \dots & 0 \end{pmatrix}.$$

By a simple computation,

$$\begin{aligned} dA &= \sum_{i < j} (\omega_{2i,2j-1} + \omega_{2i-1,2j})(e_{2j-1}e_{2i-1}^t - e_{2j}e_{2i}^t - e_{2i-1}e_{2j-1}^t + e_{2i}e_{2j}^t) \\ &\quad + \sum_{i < j} (\omega_{2i,2j} - \omega_{2i-1,2j-1})(e_{2j}e_{2i-1}^t + e_{2j-1}e_{2i}^t - e_{2i}e_{2j-1}^t - e_{2i-1}e_{2j}^t) \\ &\quad + \sum_{i=1}^n \omega^{2i}(e_{2i}e_{-1}^t - e_{-1}e_{2i}^t - e_0e_{2i-1}^t + e_{2i-1}e_0^t) \\ &\quad + \sum_{i=1}^n \omega^{2i-1}(e_{2i-1}e_{-1}^t - e_{-1}e_{2i-1}^t + e_0e_{2i}^t - e_{2i}e_0^t). \end{aligned}$$

Then

$$\begin{aligned} \alpha_{ij} &= e_{2j-1}e_{2i-1}^t - e_{2j}e_{2i}^t - e_{2i-1}e_{2j-1}^t + e_{2i}e_{2j}^t, \\ \beta_{ij} &= e_{2j}e_{2i-1}^t + e_{2j-1}e_{2i}^t - e_{2i}e_{2j-1}^t - e_{2i-1}e_{2j}^t, \\ \widetilde{X}_{2i-1} &= e_{2i-1}e_{-1}^t - e_{-1}e_{2i-1}^t + e_0e_{2i}^t - e_{2i}e_0^t, \\ \widetilde{X}_{2i} &= e_{2i}e_{-1}^t - e_{-1}e_{2i}^t - e_0e_{2i-1}^t + e_{2i-1}e_0^t \end{aligned}$$

are local tangent vector fields on $\widetilde{\mathcal{J}}(S^{2n})$. The Lemma follows from

$$A\alpha_{ij} = \beta_{ij}, \quad A\widetilde{X}_{2i-1} = \widetilde{X}_{2i}. \quad \square$$

Let $T^V \widetilde{\mathcal{J}}(S^{2n}), T^H \widetilde{\mathcal{J}}(S^{2n})$ the subspaces of $T\widetilde{\mathcal{J}}(S^{2n})$ generated by α_{ij}, β_{ij} and $\widetilde{X}_{2i-1}, \widetilde{X}_{2i}$ respectively. It is easy to see that $T^V \widetilde{\mathcal{J}}(S^{2n})$ are tangent to the fibres of fibre bundle $\pi: \widetilde{\mathcal{J}}(S^{2n}) \rightarrow S^{2n}$ and

$$\pi_*(\widetilde{X}_{2i-1}) = e_{2i-1}, \quad \pi_*(\widetilde{X}_{2i}) = e_{2i}.$$

Note that

$$\sum_{l=1}^{2n} \omega^l \widetilde{X}_l = de_0e_{-1}^t - e_{-1}de_0^t - Ade_0e_0^t + e_0de_0^t A.$$

The subspaces $T^V \widetilde{\mathcal{J}}(S^{2n})$ and $T^H \widetilde{\mathcal{J}}(S^{2n})$ are orthogonal with respect to the Riemannian metric on $\widetilde{\mathcal{J}}(S^{2n})$ and are called the vertical and horizontal subspaces of $T\widetilde{\mathcal{J}}(S^{2n})$ respectively.

By Theorem 3.1, \tilde{J} is a complex structure on $\widetilde{\mathcal{J}}(S^{2n})$ and preserves the decomposition

$$T\widetilde{\mathcal{J}}(S^{2n}) = T^V \widetilde{\mathcal{J}}(S^{2n}) \oplus T^H \widetilde{\mathcal{J}}(S^{2n}).$$

On the other hand, any $A \in \widetilde{\mathcal{J}}(S^{2n})$ defines a complex structure J_A on $T_{e_0} S^{2n}$:

$$X \in T_{e_0} S^{2n} \mapsto J_A(X) = AX, \quad e_0 = \pi(A).$$

$\pi^{-1}(e_0)$ is the set of all complex structure on $T_{e_0} S^{2n}$. By Theorem 2.3, every fibre of $\pi: \widetilde{\mathcal{J}}(S^{2n}) \rightarrow S^{2n}$ is a Kaehler submanifold of $\mathcal{J}(S^{2n})$.

We have proved the following

Lemma 3.2 (1) The following diagram of maps is commutative,

$$\begin{array}{ccc} T_A \widetilde{\mathcal{J}}(S^{2n}) & \xrightarrow{\tilde{J}} & T_A \widetilde{\mathcal{J}}(S^{2n}) \\ \pi_* \downarrow & & \downarrow \pi_* \\ T_{e_0} S^{2n} & \xrightarrow{J_A} & T_{e_0} S^{2n}; \end{array}$$

(2) $\tilde{J}: T^H \widetilde{\mathcal{J}}(S^{2n}) \rightarrow T^H \widetilde{\mathcal{J}}(S^{2n})$, $\tilde{J}: T^V \widetilde{\mathcal{J}}(S^{2n}) \rightarrow T^V \widetilde{\mathcal{J}}(S^{2n})$.

As we know, there is a one-one map between the almost complex structures on the sphere S^{2n} and the sections of fibre bundle π . Let $f: S^{2n} \rightarrow \widetilde{\mathcal{J}}(S^{2n})$ be a section, J_f be the corresponding orthogonal almost complex structure, $J_{f(v)}: T_v S^{2n} \rightarrow T_v S^{2n}$. The map f is holomorphic if $f_* J_f = \tilde{J} f_*$.

From Lemma 3.2 and $\pi \circ f = id$, we have

Lemma 3.3 $J_{f(v)} = \pi_* \tilde{J} f_*$.

For any $X \in T_v(S^{2n})$, set $f_*(X) = Z_1 + Z_2$, $Z_1 \in T_{f(v)}^H \widetilde{\mathcal{J}}(S^{2n})$, $Z_2 \in T_{f(v)}^V \widetilde{\mathcal{J}}(S^{2n})$, we have $J_{f(v)}(X) = \pi_* \tilde{J}(Z_1)$. On the other hand, the tangent vector X can be left to a horizontal vector $\tilde{X} \in T_{f(v)}^H \widetilde{\mathcal{J}}(S^{2n})$, it is easy to see that $\tilde{X} = Z_1$. The complex structure $J_{f(v)}: T_v S^{2n} \rightarrow T_v S^{2n}$ is determined by the complex structure \tilde{J} on $T_{f(v)}^H \widetilde{\mathcal{J}}(S^{2n})$ and the isomorphism $\pi_*: T_{f(v)}^H \widetilde{\mathcal{J}}(S^{2n}) \rightarrow T_v S^{2n}$.

By Theorem 3.1, $\widetilde{\mathcal{J}}(S^{2n})$ is a Kaehler manifold, then $\dim H^{2k}(\widetilde{\mathcal{J}}(S^{2n})) \geq 1$ for $k = 0, 1, \dots, n^2 + n$. $H^{2k}(\widetilde{\mathcal{J}}(S^{2n}))$ is generated by the Kaehler form on $\widetilde{\mathcal{J}}(S^{2n})$ if $\dim H^{2k}(\widetilde{\mathcal{J}}(S^{2n})) = 1$.

In the following we study the problem of the existence of complex structure on sphere S^{2n} . By [2], there is an almost complex structure on S^{2n} if and only if $n = 1, 3$. As $S^2 = CP^1$ is a Kaehler manifold, we need only consider the sphere S^6 .

Let $f: S^{2n} \rightarrow \widetilde{\mathcal{J}}(S^{2n})$ be a section of fibre bundle π , the section exists if only if $n = 1, 3$. By almost complex structure J_f , we have

$$TS^{2n} \otimes C = T^{(1,0)}S^{2n} \oplus T^{(0,1)}S^{2n},$$

where $T^{(1,0)}S^{2n} = \{X - \sqrt{-1}J_f X \mid X \in TS^{2n}\}$ and $T^{(0,1)}S^{2n} = \overline{T^{(1,0)}S^{2n}}$. The almost complex structure J_f is integrable if only if

$$[X, Y] \in \Gamma(T^{(1,0)}S^{2n}) \text{ for any } X, Y \in \Gamma(T^{(1,0)}S^{2n}).$$

Similarly, with respect to the complex structure \widetilde{J} , we have

$$T\widetilde{\mathcal{J}}(S^{2n}) \otimes C = T^{(1,0)}\widetilde{\mathcal{J}}(S^{2n}) \oplus T^{(0,1)}\widetilde{\mathcal{J}}(S^{2n}).$$

The section $f: S^{2n} \rightarrow \widetilde{\mathcal{J}}(S^{2n})$ can be viewed as a map $f: S^{2n} \rightarrow gl(2n+2, R)$ and $T_A\widetilde{\mathcal{J}}(S^{2n})$ is a subspace of $gl(2n+2, R)$, then

$$f_*X = Xf$$

for any $X \in \Gamma(TS^{2n})$ or $X \in \Gamma(TS^{2n} \otimes C)$.

On the other hand, the section f defines an almost complex structure $J_f \in \Gamma(End(TS^{2n}))$. We can compute $\nabla_X J_f$, where ∇ is the Riemannian connection on the sphere S^{2n} .

Theorem 3.4 The orthogonal almost complex structure J_f on S^{2n} is integrable if and only if the map f is holomorphic. Then there is no integrable complex structure on the sphere S^{2n} for $n > 1$.

First we prove

Lemma 3.5 $f_*X = \nabla_X J_f + \widetilde{X}$, where \widetilde{X} is the horizontal lift of X to $T^H\widetilde{\mathcal{J}}(S^{2n})$ and $\nabla_X J_f \in \Gamma(T^V\widetilde{\mathcal{J}}(S^{2n}))$ is a vertical vector field.

Proof Let e_1, \dots, e_{2n} be local orthonormal vector fields on S^{2n} . The section f can be represented by

$$f = e_0 \cdot e_{-1}^t - e_{-1} \cdot e_0^t + (e_1, \dots, e_{2n})B(e_1, \dots, e_{2n})^t,$$

where $e_0 \in S^{2n}$ and B a matrix function on S^{2n} , $B^2 = -I$ and $BB^t = I$. Let

$$de_0 = (e_1, \dots, e_{2n})(\omega^1, \dots, \omega^{2n})^t,$$

$$d(e_1, \dots, e_{2n}) = (e_1, \dots, e_{2n})\omega - e_0(\omega^1, \dots, \omega^{2n}).$$

Hence

$$\nabla(e_1, \dots, e_{2n}) = (e_1, \dots, e_{2n})\omega,$$

$$df = (e_1, \dots, e_{2n})(dB + \omega B - B\omega)(e_1, \dots, e_{2n})^t + \sum_{l=1}^{2n} \omega^l \widetilde{X}_l.$$

On other hand, as an endomorphism on TS^{2n} , J_f is represented by

$$J_f = (e_1, \dots, e_{2n})B(e_1, \dots, e_{2n})^t,$$

$$J_f(X) = (e_1, \dots, e_{2n})B(e_1, \dots, e_{2n})^t X = \sum B_{kl}e_k(e_l^t X) = \sum X^l B_{kl}e_k,$$

where $X = \sum X^l e_l \in TS^{2n}$. The Lemma follows from

$$\nabla_X J_f = (e_1, \dots, e_{2n})(XB + \omega(X)B - B\omega(X))(e_1, \dots, e_{2n})^t,$$

and

$$f_*X = Xf = \nabla_X J_f + \widetilde{X},$$

where $\widetilde{X} = \sum_{l=1}^{2n} X^l \widetilde{X}_l$ is the horizontal lift of X and $\nabla_X J_f \in \Gamma(T^V \widetilde{\mathcal{J}}(S^{2n}))$. \square

Now we complete the proof of Theorem 3.4. By Lemma 3.2, $\widetilde{X} \in \Gamma(T^{(1,0)} \widetilde{\mathcal{J}}(S^{2n}))$ for any $X \in \Gamma(T^{(1,0)} S^{2n})$. It is easy to see that the almost complex structure J_f is integrable if the map f is holomorphic.

Next assuming that the almost complex structure J_f is integrable. We show $\nabla_X J_f \in \Gamma(T^{(1,0)} \widetilde{\mathcal{J}}(S^{2n}))$ for any $X \in \Gamma(T^{(1,0)} S^{2n})$, then the map f is holomorphic.

In this case, $[X, Y] = \nabla_X Y - \nabla_Y X \in \Gamma(T^{(1,0)} S^{2n})$ for any $X, Y \in \Gamma(T^{(1,0)} S^{2n})$. As in [5], [9], we can show

$$\nabla_X Y, \nabla_Y X \in \Gamma(T^{(1,0)} S^{2n}).$$

These can also be proved as in [1]:

For any $X, Y, Z \in \Gamma(T^{(1,0)} S^{2n})$, $\langle X, Y \rangle = 0$ with respect to the standard inner product on TS^{2n} . Then $0 = Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$, hence

$$\begin{aligned} \langle \nabla_X Y, Z \rangle &= \langle \nabla_Y X, Z \rangle = -\langle X, \nabla_Y Z \rangle = -\langle X, \nabla_Z Y \rangle \\ &= \langle \nabla_Z X, Y \rangle = \langle \nabla_X Z, Y \rangle = -\langle Z, \nabla_X Y \rangle, \end{aligned}$$

and $\langle \nabla_X Y, Z \rangle = 0$ for any $Z \in \Gamma(T^{(1,0)} S^{2n})$. This shows $\nabla_X Y \in \Gamma(T^{(1,0)} S^{2n})$.

As $Y, \nabla_X Y \in \Gamma(T^{(1,0)} S^{2n})$, we have

$$(\nabla_X J_f)Y = \nabla_X(J_f Y) - J_f \nabla_X Y = 0.$$

Then for any $Y_1 \in \Gamma(S^{2n})$, $Y = (1 - \sqrt{-1}J_f)Y_1 \in \Gamma(T^{(1,0)} S^{2n})$, we have

$$(\nabla_X J_f)Y = (\nabla_X J_f - \sqrt{-1}\nabla_X J_f \cdot J_f)Y_1 = 0.$$

Then

$$\begin{aligned}\nabla_X J_f - \sqrt{-1} \nabla_X J_f \cdot J_f &= 0, \\ \nabla_X J_f &= \sqrt{-1} \nabla_X J_f \cdot J_f = -\sqrt{-1} J_f \nabla_X J_f.\end{aligned}$$

By Lemma 3.5, $\nabla_X J_f \in \Gamma(T^V \widetilde{\mathcal{J}}(S^{2n}) \otimes C)$, from $J_f \nabla_X J_f = \sqrt{-1} \nabla_X J_f$, we have

$$\widetilde{J} \nabla_X J_f = J_f \nabla_X J_f = \sqrt{-1} \nabla_X J_f.$$

These shows $Xf \in \Gamma(T^{(1,0)} \widetilde{\mathcal{J}}(S^{2n}))$ for any $X \in \Gamma(T^{(1,0)}(S^{2n}))$ and the map $f: S^{2n} \rightarrow \widetilde{\mathcal{J}}(S^{2n})$ is holomorphic.

Hence, if there is an integrable complex structure on the sphere S^{2n} , the related section $f: S^{2n} \rightarrow \widetilde{\mathcal{J}}(S^{2n})$ is holomorphic. Then S^{2n} is also a Kaehler manifold with the induced metric and the complex structure J_f . This can occur only when $n = 1$. We have proved that there is no integrable orthogonal complex structure on the sphere S^{2n} for $n > 1$. \square

There are two connected components on $\widetilde{\mathcal{J}}(S^{2n})$. Denote $\widetilde{\mathcal{J}}^+(S^{2n})$ the twistor space of all oriented almost orthogonal complex structure on S^{2n} . Using degenerate Morse functions we can show

Theorem 3.6 The Poincaré polynomial of $\widetilde{\mathcal{J}}^+(S^{2n})$ is

$$P_t(\widetilde{\mathcal{J}}^+(S^{2n})) = (1 + t^2)(1 + t^4) \cdots (1 + t^{2n}).$$

Proof As in [10],[11], let

$$h(A) = \langle A, \bar{e}_0 \bar{e}_{-1}^t - \bar{e}_{-1} \bar{e}_0^t \rangle$$

be a function on $\widetilde{\mathcal{J}}^+(S^{2n})$, where $\bar{e}_{-1} = (1, 0, 0, \dots, 0)^t$, $\bar{e}_0 = (0, 1, 0, \dots, 0)^t$ are two fixed vectors. As discussed above, we can set

$$\begin{aligned}A &= (e_{-1}, e_0, e_1, \dots, e_{2n}) J_0 (e_{-1}, e_0, e_1, \dots, e_{2n})^t, \\ (e_{-1}, e_0, e_1, \dots, e_{2n}) &\in SO(2n + 2), \quad e_{-1} = \bar{e}_{-1}.\end{aligned}$$

The elements $\{e_A e_B^t, A, B = -1, 0, 1, \dots, 2n\}$ forms an orthonormal basis of $gl(2n + 2, R)$ with the norm $\langle X, Y \rangle = \text{tr } XY^t$. Hence

$$\begin{aligned}dh &= \sum_{i=1}^n \omega^{2i} \langle e_{2i} \bar{e}_{-1}^t - \bar{e}_{-1} e_{2i}^t, \bar{e}_0 \bar{e}_{-1}^t - \bar{e}_{-1} \bar{e}_0^t \rangle \\ &\quad + \sum_{i=1}^n \omega^{2i-1} \langle e_{2i-1} \bar{e}_{-1}^t - \bar{e}_{-1} e_{2i-1}^t, \bar{e}_0 \bar{e}_{-1}^t - \bar{e}_{-1} \bar{e}_0^t \rangle \\ &= 2 \sum_{l=1}^{2n} \omega^l \langle e_l, \bar{e}_0 \rangle,\end{aligned}$$

$dh = 0$ if and only if $e_0 = \bar{e}_0$ or $e_0 = -\bar{e}_0$. Then $\pi^{-1}(\bar{e}_0)$ and $\pi^{-1}(-\bar{e}_0)$ are two critical submanifolds of the function h . It is easy to see that

$$d^2h|_{\pi^{-1}(\bar{e}_0)} = -2 \sum_{l=1}^{2n} \omega^l \otimes \omega^l, \quad d^2h|_{\pi^{-1}(-\bar{e}_0)} = 2 \sum_{l=1}^{2n} \omega^l \otimes \omega^l.$$

These shows that the critical submanifolds $h^{-1}(-2) = \pi^{-1}(-\bar{e}_0)$ and $h^{-1}(2) = \pi^{-1}(\bar{e}_0)$ are non-degenerate with indices 0 and $2n$ respectively. Then h is a Morse function and the Poincaré polynomial of $\widetilde{\mathcal{J}}^+(S^{2n})$ is

$$P_t(\widetilde{\mathcal{J}}^+(S^{2n})) = (1 + t^{2n})P_t(\widetilde{\mathcal{J}}^+(S^{2n-2})). \quad \square$$

In particular, $P_t(\mathcal{J}^+(S^4)) = P_t(\widetilde{\mathcal{J}}^+(S^4)) = 1 + t^2 + t^4 + t^6$. This also shows there is no almost complex structure on the sphere S^4 , for proof see [9].

4. The complex structure on the sphere S^{2n}

In the following we study the complex structure on the sphere S^{2n} and give another proof of Theorem 3.4.

The metric on S^{2n} can be represented by

$$ds^2 = \frac{1}{(1 + \frac{1}{4}|y|^2)^2} \sum_{i=1}^{2n} (dy^i)^2,$$

where $(y^1, y^2, \dots, y^{2n})$ are the coordinates on S^{2n} defined by the stereographic projection. Let $e_i = (1 + \frac{1}{4}|y|^2) \frac{\partial}{\partial y^i}$ be an orthonormal frame fields on S^{2n} , $\omega^i = \frac{1}{1 + \frac{1}{4}|y|^2} dy^i$ be their dual 1-forms.

Lemma 4.1 The Riemannian connection ∇ on S^{2n} is defined by

$$\nabla_{e_i} e_j = -\frac{1}{2} y^j e_i + \frac{1}{2} \sum_k y^k e_k \delta_{ij}.$$

Proof By the structure equations of Riemannian connection

$$d\omega^k = \sum_j \omega^j \wedge \omega_j^k, \quad \omega_j^k + \omega_k^j = 0,$$

we have

$$\omega_j^k = -\frac{1}{2} y^j \omega^k + \frac{1}{2} y^k \omega^j, \quad k, j = 1, 2, \dots, 2n.$$

The Riemannian connection ∇ on S^{2n} is defined by $\nabla e_j = \sum_k \omega_j^k e_k$. \square

By $\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j = -\omega^i \wedge \omega^j$, we know that the sphere has constant curvature.

In the following we sometimes omit the notation \sum .

Lemma 4.2 Let J be an almost complex structure on S^{2n} , $Je_i = \sum e_j B_{ji}$. We have

$$\begin{aligned} & \nabla_{e_i - \sqrt{-1}Je_i}(e_j - \sqrt{-1}Je_j) \\ &= -\sqrt{-1}[(e_i - \sqrt{-1}Je_i)B_{kj}]e_k - \frac{1}{2}(y^j - \sqrt{-1}y^k B_{kj})(e_i - \sqrt{-1}Je_i) \\ & \quad + \frac{1}{2}[\delta_{ij} - B_{ki}B_{kj} - \sqrt{-1}(B_{ij} + B_{ji})]y^l e_l. \end{aligned}$$

Then

$$\begin{aligned} & [e_i - \sqrt{-1}Je_i, e_j - \sqrt{-1}Je_j] \\ &= -\sqrt{-1}[(e_i - \sqrt{-1}Je_i)B_{kj}]e_k + \sqrt{-1}[(e_j - \sqrt{-1}Je_j)B_{ki}]e_k \\ & \quad - \frac{1}{2}(y^j - \sqrt{-1}y^k B_{kj})(e_i - \sqrt{-1}Je_i) + \frac{1}{2}(y^i - \sqrt{-1}y^k B_{ki})(e_j - \sqrt{-1}Je_j). \end{aligned}$$

Remark If the almost complex structure J is orthogonal, we have $BB^t = I$ and $B + B^t = 0$, then

$$\begin{aligned} & \nabla_{e_i - \sqrt{-1}Je_i}(e_j - \sqrt{-1}Je_j) \\ &= -\sqrt{-1}[(e_i - \sqrt{-1}Je_i)B_{kj}]e_k - \frac{1}{2}(y^j - \sqrt{-1}y^k B_{kj})(e_i - \sqrt{-1}Je_i). \end{aligned}$$

$X_i = e_i - \sqrt{-1}Je_i$ are $(1, 0)$ vector fields on S^{2n} , then $\nabla_{X_i}X_j \neq -\nabla_{X_j}X_i$, $\nabla_{\overline{X}_i}\overline{X}_j \neq -\nabla_{\overline{X}_j}\overline{X}_i$. But in Lebrun's paper [6], he proved $\nabla_{X_\alpha}X_\beta = -\nabla_{X_\beta}X_\alpha$ for $(0, 1)$ vector fields X_α, X_β .

Corollary 4.3 (1) The almost complex structure J is integrable if and only if $\sum [(e_i - \sqrt{-1}Je_i)B_{kj}]e_k - \sum [(e_j - \sqrt{-1}Je_j)B_{ki}]e_k$ is $(1, 0)$ for any i, j , that is

$$\sum e_k[e_i B_{kj} - e_j B_{ki}] + \sum J e_k[(J e_i)B_{kj} - (J e_j)B_{ki}] = 0;$$

(2) If J is an orthogonal almost complex structure on S^{2n} , J integrable if and only if $\sum [(e_i - \sqrt{-1}Je_i)B_{kj}]e_k$ is $(1, 0)$ for any i, j , that is

$$\sum e_k(e_i B_{kj}) + \sum J e_k(J e_i)B_{kj} = 0.$$

Proof (1) By Lemma 4.2, $[e_i - \sqrt{-1}Je_i, e_j - \sqrt{-1}Je_j]$ is $(1, 0)$ if and only if

$$\sum [(e_i - \sqrt{-1}Je_i)B_{kj}]e_k - \sum [(e_j - \sqrt{-1}Je_j)B_{ki}]e_k$$

is $(1, 0)$. Act J on the left of above equation we have

$$\sum [(e_i B_{kj})e_k - (e_j B_{ki})e_k] = - \sum [((J e_i)B_{kj})J e_k + ((J e_j)B_{ki})J e_k].$$

(2) follows from Lemma 4.2 and the fact that $\nabla_{e_i - \sqrt{-1}J e_i}(e_j - \sqrt{-1}J e_j)$ is $(1, 0)$ when the complex structure J is orthogonal. \square

Define $J\omega^i = \sum B_{ij}\omega^j$, we have $(J\omega^i)e_k = \omega^i(Je_k)$. The torsion of the complex structure J is defined by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y].$$

From $[e_i - \sqrt{-1}J e_i, e_j - \sqrt{-1}J e_j]$, we have

$$N(e_i, e_j) = \sum e_k [(Je_i)B_{kj} - (Je_j)B_{ki}] - \sum J e_k [e_i B_{kj} - e_j B_{ki}].$$

Hence the torsion of J can be represented by

$$\begin{aligned} N &= \sum [(Je_i)B_{kj} - (e_i B_{lj})B_{kl}] \omega^i \wedge \omega^j \otimes e_k \\ &= \sum e_i B_{kj} [J\omega^i \wedge \omega^j + \omega^i \wedge J\omega^j] \otimes e_k \\ &= \sum (e_i B_{kj} - e_j B_{ki}) \omega^i \wedge J\omega^j \otimes e_k. \end{aligned}$$

Note that

$$JN = \sum e_i B_{kj} [\omega^i \wedge \omega^j - J\omega^i \wedge J\omega^j] \otimes e_k.$$

Let $f = \tilde{J} = e_0 \cdot e_{-1}^t - e_{-1} \cdot e_0^t + (e_1, \dots, e_{2n})B(e_1, \dots, e_{2n})^t$ be a section of twistor space $\mathcal{J}(S^{2n})$, $B^2 = -I$. We also view f as a map from S^{2n} to the Kaehler manifold $\mathcal{J}(R^{2n+2})$. Then $J = J_f = \sum e_i B_{ij} e_j^t$ is an almost complex on S^{2n} , set

$$f_* X = Xf = \nabla_X J_f + \widehat{X}.$$

We have

$$\begin{aligned} \nabla_{e_l} J_f &= e_l (e_l B_{ij}) e_j^t + (\nabla_{e_l} e_i) B_{ij} e_j^t + e_i B_{ij} (\nabla_{e_l} e_j)^t \\ &= e_l (e_l B_{ij}) e_j^t + \frac{1}{2} (-e_l y^i e_i^t + y^k e_k e_l^t) J + \frac{1}{2} J (e_l y^i e_i^t - y^i e_i e_l^t), \end{aligned}$$

and

$$\widehat{e_l} = e_l e_{-1}^t - e_{-1} e_l^t + \tilde{J} (e_l e_{-1}^t - e_{-1} e_l^t) \tilde{J}.$$

Then

$$\begin{aligned} &\nabla_{e_l - \sqrt{-1}J e_l} J_f \\ &= e_i [(e_l - \sqrt{-1}J e_l) B_{ij}] e_j^t + \frac{\sqrt{-1}}{2} (1 - \sqrt{-1}J) e_l y^j e_j^t (1 + \sqrt{-1}J) \\ &\quad - \frac{\sqrt{-1}}{2} y^i (1 - \sqrt{-1}J) e_i e_l^t (1 + \sqrt{-1}J) \\ &\quad + \frac{\sqrt{-1}}{2} (B_{lk} + B_{kl}) [y^i J e_i e_k^t - y^i e_i e_k^t J], \end{aligned}$$

and

$$\begin{aligned} e_l - \widehat{\sqrt{-1}J}e_l &= (1 - \sqrt{-1}J)e_l e_{-1}^t (1 + \sqrt{-1}\tilde{J}) - (1 - \sqrt{-1}\tilde{J})e_{-1}(e_l^t - \sqrt{-1}B_{kl}e_k^t) \\ &\quad - e_0(B_{lk} + B_{kl})e_k^t(1 - \sqrt{-1}J). \end{aligned}$$

If the complex structure J_f is orthogonal and integrable, $B_{ij} + B_{ji} = 0$, $Xf = \nabla_X J_f + \widehat{X}$ is a $(1, 0)$ vector field on $\widetilde{\mathcal{J}}(S^{2n})$ for any $X \in \Gamma(T^{(1,0)}S^{2n})$. These gives another proof of Theorem 3.4.

The above computation can be used to show that the twistor space $\mathcal{J}(S^{2n})$ is not a Kaehler submanifold of $\mathcal{J}(R^{2n+2})$.

Theorem 4.4 Let $f: S^{2n} \rightarrow \mathcal{J}(S^{2n})$ be a local section. The map $f: S^{2n} \rightarrow \mathcal{J}(R^{2n+2})$ is holomorphic if and only if f is a local section of $\widetilde{\mathcal{J}}(S^{2n})$ and integrable.

Proof For any tangent vector e_l defined above, we have

$$\begin{aligned} (\tilde{J}f_* - f_*J_f)e_l &= J_e(e_l B_{ij})e_j^t - e_i((J_e l)B_{ij})e_j^t \\ &\quad - \tilde{J}(e_0 - \frac{1}{2}y^i e_i)(B_{lj} + B_{jl})e_j^t + (e_0 - \frac{1}{2}y^i e_i)(B_{jl} + B_{lj})e_j^t J. \end{aligned}$$

This shows the local section f is holomorphic if and only if $B_{lj} + B_{jl} = 0$ and $J_e(e_l B_{ij}) - e_i((J_e l)B_{ij}) = 0$. That is, the complex structure J_f is orthogonal and integrable.

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